the barrier and along the axis of the channel; τ_C, temperature of the channel surface; φ, additional functions determined from (10). Subscripts: i, number of series terms; j, number of outside layer of the barrier, calculated from the middle layer; 1 and 2, external sides of the barrier.

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MODIFICATION OF A FINITE-ELEMENT METHOD TO CALCULATE TEMPERATURE FIELDS AVERAGED OVER ONE COORDINATE

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We examined the approximate solution of an averaged nonsteady boundary-value problem of heat conduction in a two-dimensional region bounded by two continuously differentiable curves.

When we study nonsteady thermal processes, we encounter a need to calculate the temperature field in a two-dimensional region of complex configuration. Difficulties in the solution in the general formulation of the problem lead to a need to develop methods of simplifying the original boundary-value problem. For thermotechnical thin bodies, given a small temperature drop in one of the directions, simplification of the problem is possible by making a transition to temperatures averaged in the appropriate direction. Such a situation arises in the calculation of temperature fields in thin shelves, channels, etc. It is possible, in this case, to simplify the computational procedure involved in studying the dynamics of thermal processes in a region of complex geometry.

The average problem dealt with in this study can be solved by the method of finite elements [1].

Let us examine a two-dimensional region Ω, bounded by two continuously differentiable curves $x_1 = a(y)$, $x_2 = b(y)$, $0 \leq y \leq d$, $0 < a(y) < b(y)$ (see Fig. 1). We will assume that at the initial instant of time the temperature $\theta_0(x, y)$ of the region is higher than the temperature θ_m of the medium. The transfer of heat from the side surface S of a cylinder, whose cross section is the region Ω, follows the law

$$-\lambda \left.\frac{\partial\theta}{\partial n}\right|_S = \alpha\left(\left.\theta\right|_S - \theta_m\right).$$

The original boundary-value problem with boundary conditions of the IIIrd kind has the form:

$$C\gamma \frac{\partial\theta}{\partial\tau} = \lambda\left(\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2}\right) + f_{sou}(x, y, \tau), \tag{1}$$

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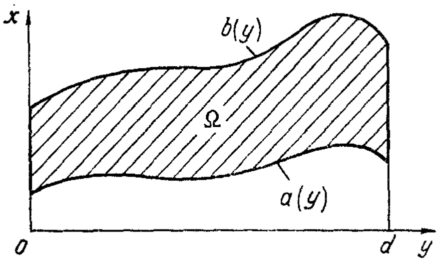

Fig. 1. Two-dimensional region with selected coordinate system.

$$\lambda\left(\frac{\partial\theta}{\partial x}\frac{1}{\sqrt{1+a'^2}}-\frac{\partial\theta}{\partial y}\frac{a'}{\sqrt{1+a'^2}}\right)=\alpha(\theta-\theta_m),\ x=a(y), \tag{2}$$

$$-\lambda\left(\frac{\partial\theta}{\partial x}\frac{1}{\sqrt{1+b'^2}}-\frac{\partial\theta}{\partial y}\frac{b'}{\sqrt{1+b'^2}}\right)=\alpha(\theta-\theta_m),\ x=b(y), \tag{3}$$

$$\lambda\frac{\partial\theta}{\partial y}=\alpha(\theta-\theta_c),\ y=0, \tag{4}$$

$$-\lambda\frac{\partial\theta}{\partial y}=\alpha(\theta-\theta_c),\ y=d, \tag{5}$$

$$\theta(x,\ y,\ 0)=\theta_0(x,\ y). \tag{6}$$

Let the temperature difference in the direction of x be sufficiently small, which is the case, for example, when $\max\limits_{0\leqslant y\leqslant d}$ (b(y) − a(y) ≪ d.

Let us introduce the function of the average temperature θ*(x, y) by averaging the temperature function in the direction of the x coordinate. This procedure is analogous to the volumetric averaging dealt with in [2].

We will apply the averaging operator

$$U[\theta]=\frac{1}{b-a}\int_a^b\theta(x,\ y,\ \tau)\,dx=\theta^*(y,\ \tau)$$

to both parts of Eq. (1):

$$C\gamma U\left[\frac{\partial\theta}{\partial\tau}\right]=\lambda\left(U\left[\frac{\partial^2\theta}{\partial x^2}\right]+U\left[\frac{\partial^2\theta}{\partial y^2}\right]\right)+U[f_{sou}].$$

Having expressed the function U[(∂²θ)/(∂y²)] in terms of the derivatives θ* and having averaged the remaining terms of Eq. (1), we obtain:

$$C\gamma\frac{\partial\theta^*}{\partial\tau}=\lambda\left(\frac{1}{b-a}\left(\frac{\partial\theta}{\partial x}\bigg|_{x=b}-\frac{\partial\theta}{\partial x}\bigg|_{x=a}\right)+\frac{\partial^2\theta^*}{\partial y^2}+\right.$$

$$\left.\frac{2(b'-a')}{b-a}\frac{\partial\theta^*}{\partial y}+\frac{b''-a''}{b-a}\theta^*-\frac{1}{b-a}\left(b'\frac{\partial\theta}{\partial y}\bigg|_{x=b}-a'\frac{\partial\theta}{\partial y}\bigg|_{x=a}\right)-\frac{1}{b-a}\frac{\partial}{\partial y}\left(b'\theta\bigg|_{x=b}-a'\theta\bigg|_{x=a}\right)\right)+f_{sou}(y,\ \tau).$$

If we assume that the temperature gradient along the coordinate is small, with consideration of (2) and (3), we have

$$C\gamma\frac{\partial\theta^*}{\partial\tau}=\lambda\left(\frac{\partial^2\theta^*}{\partial y^2}+\frac{b'-a'}{b-a}\frac{\partial\theta^*}{\partial y}\right)-\alpha\frac{\sqrt{1+b'^2}+\sqrt{1+a'^2}}{b-a}(\theta^*-\theta_m)+f^*_{sou}(y,\ \tau). \tag{7}$$

Boundary conditions (4) and (5), as well as the original condition (6), respectively, assume the form:

$$\lambda\frac{\partial\theta^*}{\partial y}=\alpha(\theta^*-\theta_m),\ y=0, \tag{8}$$

$$-\lambda\frac{\partial\theta^*}{\partial y}=\alpha(\theta^*-\theta_m),\ y=d, \tag{9}$$

$$\theta^*(y,0)=\theta_0^*(y). \tag{10}$$

The solution of the derived equation by numerical methods calls for a lower volume of calculations than does the original Eq. (1). In this case, we take into consideration information regarding the influence exerted by the shape of the regional boundary and the amount of heat transferred to the side surface insofar as these pertain to the change in temperature in the direction of the nonzero gradient.

Without disrupting generality, let us examine problem (7)-(10) in the region for which $a(y) = 0$, $0 < b_0 \le b(y) \le b_1$.

Turning to the dimensionless variables $\theta = (\theta^* - \theta_m)/\theta_m$, $t = \lambda\tau/C_\gamma d^2$, we derive the following equations:

$$\frac{\partial\theta}{\partial t} - \frac{\partial^2\theta}{\partial y^2} - \frac{b'(y)}{b(y)}\frac{\partial\theta}{\partial y} + B_0\frac{1+\sqrt{1+b'^2}}{b(y)}\theta = \tilde{f}(y, t), \tag{11}$$

$$\frac{\partial\theta}{\partial y} - B_0\theta = 0,\ y = 0, \tag{12}$$

$$\frac{\partial\theta}{\partial y} + B_0\theta = 0,\ y = 1, \tag{13}$$

$$\theta(y, 0) = \theta_0(y). \tag{14}$$

Here $B_0 = \alpha d/\lambda$, $\tilde{f}(y, t) = f_{sou}{}^*(y, t)d^2/\theta_m\lambda$.

Let us write this problem in operator form:

$$\frac{d\theta}{dt} + A\theta = f, \tag{15}$$

$$\theta(0) = \theta_0, \tag{16}$$

where $\tilde{f}(y, t) = L_2(0, 1) \times (0, T))$, $\theta_0(y) \in L_2(0, 1)$; A, is the operator $-(\partial^2\theta/\partial y^2) - (b'/b) \times (\partial\theta/\partial y) + B_0(1 + \sqrt{1+b'^2}/b)\theta$ with the determination region

$$D(A) = \{v : v \in L_2(0, 1), \frac{dv}{dy} \in L_2(0, 1),\ Av \in L_2(0, 1),$$

$$\frac{dv}{dy}(0) - B_0 v(0) = 0, \frac{dv}{dy}(1) + B_0 v(1) = 0\},\ b(y) \in C^1(0, 1),\ |b'(y)| \leqslant b_2.$$

For an approximate solution of the formulated problem we will use the projection-grid method of solving nonsteady problems [1]. The problem is solved in two stages: initially we have the approximation with respect to the spatial variable, and then with respect to time.

For the approximation with respect to y we choose a system of piecewise-linear functions $\{\varphi_i(y)\}_{i=0}^N$, constructed on a grid of nodes [1] $y_0 < y_1 < \ldots < y_{N-1} < y_N$, $y_0 = 0$, $y_N = 1$. This system exhibits the property of uniform linear independence and approximates the functions from the space $W_2{}^2(0, 1)$. The set of linear combinations of the form $v_N = \sum_{i=0}^{N} a_i\varphi_i(y)$ forms a subspace in the space $W_2{}^1(0, 1)$, and the sequence $\{H_N\}_{N=1}^\infty$ is dense to the limit in $W_2{}^1(1, 0)$. If the energy space generated by the operator A is equivalent to the space $W_2{}^1(0, 1)$, the approximate solution $u_N = \sum_{i=0}^{N} a_i\varphi_i(y)$, constructed in accordance with the Ritz method in energy spaces, will converge as $N \to \infty$ to the generalized solution u_0 in the metric of the space H_A.

We will seek the approximate solution of problem (15), (16) in the form

$$u_N(y, t) = \sum_{i=0}^{N} a_i(t)\varphi_i(y),$$

where the coefficients a_i are determined from the following system of ordinary differential equations:

$$\left(\frac{\partial u_N}{\partial t}, \varphi_i\right)(t) + [u_N, \varphi_i](t) = (\tilde{f}, \varphi_i) \tag{17}$$

under the initial conditions

$$(u_N(y, 0) - u_0, \varphi_i) = 0,\ i = \overline{0, N}, \tag{18}$$

which arises in the determination of the generalized solution. Here the parentheses denote the scalar product in the original Hilbert space $L_2(0, 1)$, while the brackets denote the scalar product in the energy space H_A.

Problem (17), (18) can be written in the form:

$$B\frac{da}{dt}+Aa=F(t), \tag{19}$$

$$B\,a(0)=a_0, \tag{20}$$

where

$$a(t)=(a_0(t),\ \ldots,\ a_N(t))^T,\quad a_0=(a_{00},\ \ldots,\ a_{0N})^T,\quad a_{0i}=(\theta_0,\ \varphi_i),\quad i=\overline{0,\ N};\ F(t)=(F_0(t),\ \ldots,\ F_N(t))^T,\quad F_i(t)=(\tilde{f},\ \varphi_i),\ i=\overline{0,\ N};\quad B=(B_{ij}),$$
$$B_{ij}=B_{ji}=(\varphi_i,\ \varphi_j),\ i,\ j=\overline{0,\ N};\quad A=(A_{ij}),\quad A_{ij}=A_{ji}=[\varphi_i,\ \varphi_j],\quad i,\ j=\overline{0,\ N}.$$

If the original operator A exhibits the property of positive determinacy, system (19), (20) will have the singular solution

$$a(t)=\exp\{-B^{-1}At\}B^{-1}a_0+\int_0^t\exp\{-B^{-1}A(t-t')\}B^{-1}F(t')\,dt',$$

where $\exp\{Gt\}$ is the matrix exponent.

Thus, the approximate solution of problem (15), (16) will be determined uniquely.

Let us now turn to validation of the possibility of using this method in connection with the solution of the original problem (11)-(14).

Let us examine the Hilbert space $L_2{}^b(0,\ 1)$ with the scalar product

$$(u,\ v)_{L_2^b(0,1)}=\int_0^1 u(y)\,v(y)\,b(y)\,dy.$$

It is obvious that the spaces $L_2(0,\ 1)$ and $L_2{}^b(0,\ 1)$ are equivalent, since b(y) assumes only positive values and is bounded. Consequently, determination of the generalized solution has significance even when $L_2{}^b(0,\ 1)$ and the system of approximating functions $\{\varphi_i(y)\}_{i=0}{}^\infty$ retains its properties in $L_2{}^b(0,\ 1)$. In this space the operator of problem (15), (16) exhibits its properties of symmetricity and positive definiteness.

LEMMA 1. Operator A is symmetrical in $L_2{}^b(0,\ 1)$.

Proof. Let us examine the scalar product

$$(Au,\ v)=\int_0^1\left(-\frac{d^2u}{dy^2}-\frac{b'}{b}\frac{du}{dy}+B_0\frac{1+\sqrt{1+b'^2}}{b}u\right)vbdy=$$
$$=-\int_0^1\frac{d}{dy}\left(b\frac{du}{dy}\right)vdy+B_0\int_0^1(1+\sqrt{1+b'^2})\,uvdy=$$
$$=B_0(b(1)u(1)v(1)+b(0)u(0)v(0))+\int_0^1 b\frac{du}{dy}\frac{dv}{dy}dy+$$
$$+B_0\int_0^1(1+\sqrt{1+b'^2})\,uvdy=(v,\ Au).$$

Consequently, $A = A^*$.

LEMMA 2. Operator A is positive definite in $L_2{}^b(0,\ 1)$.

Proof. Assuming u = v, we obtain

$$(Au,\ u)=B_0(b(1)u^2(1)+b(0)u^2(0))+\int_0^1 b\left(\frac{du}{dy}\right)^2dy+$$
$$+B_0\int_0^1(1+\sqrt{1+b'^2})\,u^2dy\geqslant B_0b(0)u^2(0)+b_0\int_0^1\left(\frac{du}{dy}\right)^2dy+2B_0\|u\|^2_{L_2}\geqslant$$
$$\geqslant\tilde{\gamma}\left(u^2(0)+\int_0^1\left(\frac{du}{dy}\right)^2dy\right)+2B_0\|u\|^2_{L_2}\geqslant c_M\|u\|^2_{L_2^b},$$

and here $\tilde{\gamma} = \min\{b_0, B_0 b_0\}$, $c_m = (\gamma/2 + 2B_0)/b_1$.

In the proof we made use of the inequality

$$2\left(u^2(0) + \left\|\frac{du}{dy}\right\|^2_{L_2}\right) \geqslant \|u\|^2_{L_2}$$

and the equality of the spaces $L_2{}^b(0, 1)$ and $L_2(0, 1)$.

Let us now turn to an examination of the energy space generated by operator A. For this we will introduce the scalar product of two functions from $L_2{}^b(0, 1)$, having the following form:

$$[u, v] = \int_0^1 \left(b\frac{du}{dy}\frac{dv}{dy} + B_0(1 + \sqrt{1 + b'^2})uv\right)dy + B_0(b(1)u(1)v(1) + b(0)u(0)v(0)).$$

The space specified by such a scalar product will be known as the energy space H_A.

LEMMA 3. Spaces H_A and $W_2{}^1(0, 1)$ are equivalent.

Proof. By definition,

$$[u, u] = \int_0^1 b\left(\frac{du}{dy}\right)^2 dy + B_0\int_0^1 (1 + \sqrt{1 + b'^2})u^2 dy + B_0(b(1)u^2(1) + b(0)u^2(0)) \leqslant$$

$$\leqslant b_1 \int_0^1 \left(\frac{du}{dy}\right)^2 dy + B_0 b_3 \int_0^1 u^2 dy + B_0 b_1 (u^2(1) + u^2(0)) \leqslant$$

$$\leqslant c_1 \int_0^1 \left(\left(\frac{du}{dy}\right)^2 + u^2\right)dy + B_0 b_1 c_2 \int_0^1 \left(\left(\frac{du}{dy}\right)^2 + u^2\right)dy \leqslant c_3 \int_0^1 \left(\left(\frac{du}{dy}\right)^2 + u^2\right)dy = c_3\|u\|^2_{W_2^1},$$

and here $b_3 = \max\limits_{0 \leqslant y \leqslant 1}(1 + \sqrt{1 + b'^2})$, $c_1 = \max\{b_1, B_0 b_3\}$, c_2 is a constant from [1] the inequality $\|u\|_{L_2(\partial\Omega)}{}^2 \leq c_2 \|u\|_{W_2{}^1(\Omega)}{}^2$.

On the other hand,

$$\|u\|^2_{W_2^1} = \int_0^1 \left(\left(\frac{du}{dy}\right)^2 + u^2\right)dy = \frac{1}{b_0}\int_0^1 b_0\left(\frac{du}{dy}\right)^2 dy + \frac{1}{2B_0}\int_0^1 2B_0 u^2 dy \leqslant$$

$$\leqslant c_4 \int_0^1 \left(b\left(\frac{du}{dy}\right)^2 + B_0(1 + \sqrt{1 + b'^2})u^2\right)dy + c_4 B_0(b(1)u^2(1) + b(0)u^2(0)) = c_4\|u\|^2_{H_A},$$

and here $c_4 = \max\{(1/b_0), (1/2B_0\}$.

Consequently, the approximation of the solution by means of the selected basis functions is possible and the solution of the problem will be a function from $W_2{}^1(0, 1)$.

If the temperature of the ambiant medium changes over time, then $\theta_m = \theta_m(\tau)$ and the temperature of the ambiant medium at the initial instant of time is chosen as the characteristic parameter. Changing over to dimensionless variables, we obtain the following boundary-value problem

$$\frac{\partial\theta}{\partial y} = \frac{\partial^2\theta}{\partial y^2} + \frac{b'(y)}{b(y)}\frac{\partial\theta}{\partial y} - B_0\frac{1 + \sqrt{1 + b'^2}}{b(y)}\theta + \bar{f}(y, t), \tag{21}$$

$$\frac{\partial\theta}{\partial y} - B_0\theta = -B_0\theta_m, \quad y = 0, \tag{22}$$

$$\frac{\partial\theta}{\partial y} + B_0\theta = B_0\theta_m, \quad y = 1, \tag{23}$$

$$\theta(0, y) = \theta_0. \tag{24}$$

Here

$$\theta = \theta^*/\theta_m(0), \quad \bar{f}(y, t) = \frac{f^*_{\text{sou}}(y, t)d^2}{\lambda\theta_m(0)} + \frac{\alpha d}{\lambda}\theta_m(t)\frac{1 + \sqrt{1 + b'^2}}{b(y)}.$$

In this case the region for the determination of the operator

$$D(A)=\left\{v: v\in L_2(0,\ 1),\ \frac{dv}{dy}\in L_2(0,\ 1),\ \frac{dv}{dy}(0)-B_0v(0)=-B_0\theta_m,\right.$$
$$\left.\frac{dv}{dy}(1)+B_0v(1)=B_0\theta_m\right\},$$

and the scalar product (Au, v) in the space $L_2{}^b(0,\ 1)$ has the form:

$$(Au,\ v)=\int_0^1\left(-\frac{d^2u}{dy^2}-\frac{b'(y)}{b(y)}\frac{du}{dy}+B_0\frac{1+\sqrt{1+b'^2}}{b(y)}u\right)vbdy=$$
$$=-b(1)v(1)\frac{du}{dy}(1)+b(0)v(0)\frac{du}{dy}(0)+\int_0^1\left(b\frac{du}{dy}\frac{dv}{dy}+\right.$$
$$\left.+B_0(1+\sqrt{1+b'^2})uv\right)dy=[u,\ v]-\theta_c(t)B_0(b(1)v(1)+b(0)v(0)),$$

where [u, v] is the earlier scalar product specified by operator A with uniform boundary conditions.

Consequently, Eq. (15), on multiplication by the function from $L_2{}^b(0,\ 1)$, is written as

$$\left(\frac{du}{dt},\ v\right)+[u,\ v]=(\tilde{f},\ v)+B_0\theta_m(t)(b(1)v(1)+b(0)v(0)).$$

Thus, the nonuniformity of the boundary conditions over time will be taken into consideration in the second stage of the approximation of the solution over time.

Having chosen the system of basis functions $\{\varphi_i(y)\}_{i=0}^{\infty}$ and applying all of the considerations for Eq. (21), we come to the conclusion that in this case the equation for the determination of the coefficients of expansion over the basis functions $a_i(t)$ has the following form:

$$B\frac{da}{dt}+Aa=F(t)+B_0\theta_m(t)\Phi_{[0,1]},$$

where the matrices A, B are the same as in Eq. (19); $\Phi_{[0,1]}$ is the vector column of the form $(b(0)\varphi_0(0),\ 0,\ \ldots,\ 0,\ b(1)\varphi_N(1))^T$, which is associated with the change in temperature at the ends of the segment (0, 1); F(t) is the vector column of elements $(\tilde{f},\ \varphi_i)$, where $\tilde{f}$ is the function in the right-hand side of Eq. (21).

In conclusion, let us note that the a priori estimates of the convergence are analogous to those obtained in [1], since the norms in $L_2(0,\ 1)$ and $L_2{}^b(0,\ 1)$ are equivalent:

$$\max_t\|u-u_N\|_{L_2^b}\leqslant c^*h^{3/2},\quad \int_0^T\|u-u_N\|^2_{L_2^b}\leqslant c^{**}h^2.$$

NOTATION

λ, coefficient of thermal conductivity; α, heat-transfer coefficient; n, unit normal to the surface; C, heat capacity of the material; γ, specific weight; $f_{sou}(x,\ y,\ \tau)$, distribution density for the internal sources of heat; $f_{sou}{}^*(y,\ \tau)$, specific power of internal heat sources averaged over the excoordinate.

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